Lecture Notes

Special Functions

1. HERMITE DIFFERENTIAL EQUATION

French Mathematician Charles Hermite (1822-1901), an inspiring teacher is renowned for his proof of the transcendental character of e and solution of differential equation.

The second order homogeneous differential equation of the form

$$\boldsymbol{\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (\lambda - 1)y = 0 \ldots\ldots\ldots(1)}$$

where λ is a constant is known as Hermite differential equation. When λ is an odd integer i.e., when $\lambda = 2n + 1; n = 0,1,2 \ldots$ ....then one of the solutions of equation (1) becomes a polynomial. These polynomial solutions are known as *Hermite Polynomial* denoted by $H_n(x)$. Hermite polynomials appear in many diverse areas, the most important being in the solutions of the simple wave functions of hydrogen atom.

2. SOLUTION OF HERMITE DIFFERENTIAL EQUATION

Hermite differential equation does not have any singularity in the finite plane. So, we shall use the Power Series method to solve Hermite differential equation as given by

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (\lambda - 1)y = 0 \ldots\ldots\ldots(1)$$

where λ is a constant given by $\lambda = 2n + 1$

Equation (1) takes the form

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0 \ldots\ldots\ldots(2)$$

Here n is a non-negative constant

The various steps followed to solve the above equation are discussed below:

Step-I:

Suppose the series solution of equation (1) as

$$y(x) = \sum_{r=0}^{\infty} a_r\, x^{k+r} \ldots\ldots(2a)$$

$$\Rightarrow y'(x) = \sum_{r=0}^{\infty} a_r\,(k+r)x^{k+r-1} \ldots\ldots(2b)$$

$$\Rightarrow y''(x) = \sum_{r=0}^{\infty} a_r\,(k+r)(k+r-1)x^{k+r-2} \ldots\ldots(2c)$$

Using the equations (2a) – (2c) in equation (2), we obtain;

$$\sum_{r=0}^{\infty} a_r(k+r)\,(k+r-1)x^{k+r-2} - 2x\sum_{r=0}^{\infty} a_r\,(k+r)x^{k+r-1} + 2n\sum_{r=0}^{\infty} a_r\,x^{k+r} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r\,(k+r)(k+r-1)x^{k+r-2} - 2\sum_{r=0}^{\infty} a_r\,(k+r-n)x^{k+r} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} [(k+r)(k+r-1)x^{k+r-2} - 2(k+r-n)x^{k+r}]a_r = 0 \ldots\ldots(3)$$

Equation (3) is an identity and that is why the coefficients of various powers of x must be zero.

Step – II: Setting up of Recursion Relation

Equating the coefficient of the lowest power of x i.e. x^{k-2} (putting $r = 0$) to zero, we get

$$a_0 k(k-1) = 0$$

$$i.e., \boldsymbol{k = 0}; \qquad \boldsymbol{k = 1}$$

(As $a_0 \neq 0$, as it is the first term of the series)

Again equating the coefficient of x^{k-1} to zero (putting $r = 1$), we get

$$a_1(k+1)k = 0$$

As $k = 0$ so, from the above relation, we can write, $a_1 = 0$

Now equating the coefficient of general term x^{k+r} to zero, we obtain,

$$a_{r+2}(k+r+2)(k+r+1) - 2a_r(k+r-n) = 0$$

$$\Rightarrow \boldsymbol{a_{r+2} = \frac{2(k+r-n)}{(k+r+2)(k+r+1)} a_r} \ldots\ldots\ldots(\mathbf{4})$$

This is the recursion or recurrence relation between the coefficients.

Step-III: Determination of values of coefficients

Case-A: $\boldsymbol{k = 0}$

For $k = 0$, the recursion relation given by equation (4) takes the form:

$$a_{r+2} = \frac{2(r-n)}{(r+2)(r+1)} a_r \ldots\ldots\ldots(4a)$$

Putting $r = 0, 1, 2, 3, \ldots\ldots\ldots$, we get

$$a_2 = \frac{-2n}{2.1} a_0 = \frac{-2n}{2!} a_0$$

$$a_3 = \frac{2(1-n)}{3.2.1} a_1 = \frac{-2(n-1)}{3!} a_1 = 0; since\ a_1 = 0$$

$$a_4 = \frac{2(2-n)}{4.3} a_2 = \frac{-2(n-2)}{4.3} . \frac{-2n}{2!} a_0 = \frac{2^2\,(n-2)}{4.3.2.1} a_0$$

$$a_5 = \frac{3(2-n)}{5.4} a_3 = 0$$

.

.

Thus the general terms of the coefficients are given by

$$a_{2k} = \frac{(-2)^k\, n(n-2) \ldots\ldots(n-2k+2)}{(2k)!} a_0$$

$$a_{2k+1} = \frac{(-2)^k\, n(n-2) \ldots\ldots(n-2k+1)}{(2k+1)!} a_0$$

Therefore the general solution for the case $k = 0$ is given by,

$$y(x) = \sum_{r=0}^{\infty} a_r\, x^{k+r} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$\Rightarrow y(x) = a_0\left[1 - \frac{2n}{2!}x^2 + \frac{2^2n(n-2)}{4!}x^4 - \cdots + \frac{(-2)^k n(n-2)\ldots(n-2k+2)}{(2k)!}x^{2k}\right] + a_1\left[x - \frac{2(n-1)}{3!}x^3 + \frac{2^2(n-1)(n-3)}{5!}x^5 + \cdots\right]$ (for $a_1 \neq 0$)

$y(x) = a_0\left[1 - \frac{2n}{2!}x^2 + \frac{2^2n(n-2)}{4!}x^4 - \cdots + \frac{(-2)^k n(n-2)\ldots(n-2k+2)}{(2k)!}x^{2k}\right] = u(x)$ (say)….(5a)
(for $a_1 = 0$)

Case-B: For $k = 1$

In this case, the recursion relation (4) takes the form:

$$a_{r+2} = \frac{2(1+r) - 2n}{(r+3)(r+2)}a_r \ldots\ldots(4b)$$

Putting $r = 1, 3, 5 \ldots\ldots$ in the above equation, we get,
$a_3 = a_5 = a_7 = \cdots = 0\ since\ \ a_1 = 0$
Also putting, $r = 0, 2, 4 \ldots\ldots$, we obtain

$$a_2 = \frac{2-2n}{3.2}a_0 = \frac{2(1-n)}{3!}a_0$$

$$a_4 = \frac{6-2n}{5.4}a_2 = \frac{2(3-n)}{5.4}.\frac{2(1-n)}{3!}a_0 = \frac{2^2(n-1)(n-3)}{5!}a_0$$

And so on.
So the general solution in this case will be,
$y(x) = a_0\left[x - \frac{2(n-1)}{3!}x^3 + \frac{2^2(n-1)(n-2)}{5!}x^5 - \cdots + \frac{(-1)^k(n-1)(n-3)\ldots(n-2k+1)}{(2k+1)!}x^{2k+1}\right] = v(x)$
(say)….(5b)

Here, we have seen that that the solution (5b) is a part of solution (5a). So, $a_1 = 0$ and $k = 0$; the solution of equation (1) can therefore be expressed as the superposition of equations (5a) and (5b). So, the general solution of Hermite differential equation is given by,

$$\boldsymbol{y(x) = Au(x) + Bv(x)} \ldots\ldots.(\boldsymbol{6})$$

Here A and B are two arbitrary constants.

3. HERMITE POLYNOMIAL $H_n(x)$

As stated earlier, Hermite polynomials $H_n(x)$ appear in diverse areas of physics, the most important of which is the harmonic oscillator problem in quantum mechanics.
The Hermite Polynomial $H_n(x)$ of order n can be expressed as

$$\boldsymbol{H_n(x) = \sum_{r=0}^{n/2}(-1)^r\frac{n!}{r!\,(n-2r)!}(2x)^{n-2r}, n = even \ldots\ldots(1a)}$$

$$\boldsymbol{H_n(x) = \sum_{r=0}^{\frac{n}{2}-1}(-1)^r\frac{n!}{r!\,(n-2r)!}(2x)^{n-2r}, n = even \ldots\ldots.(1b)}$$

(for proof see supplementary examples 6.3.1.)

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4. GENERATING FUNCTION FOR HERMITE POLYNOMIAL $H_n(x)$

The generating function for Hermite Polynomial is defined as

$$g(x,t) = e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) \dots \dots . (1)$$

Proof:
We have,

$$g(x,t) = e^{2xt-t^2} = e^{2x} . e^{-t^2} = \sum_{r=0}^{\infty} \frac{(2xt)^r}{r!} . \sum_{s=0}^{\infty} \frac{(-t^2)^s}{s!}$$

$$\Rightarrow g(x,t) = \sum_{r,s=0}^{\infty} \frac{(2x)^r}{r!\, s!} . t^{r+2s}$$

Thus the coefficient of t^n(for a fixed value of s) is given by,

$$(-1)^s . \frac{(2x)^{n-2s}}{(n-2s)!\, s!}$$

Putting $r + 2s = n$

The total coefficients of t^n will however be obtained by adding all values for possible s. Since $r = n - 2s \geq 0 \Rightarrow s \leq \frac{n}{2}$

Now, if n is even then s ranges from *0* to $\frac{n}{2}$ and if n is odd then s ranges from *0* to $\frac{n}{2} - 1$.

So, the desired coefficient of t^n is expressed as;

$$\sum_{s=0}^{n/2} (-1)^s \frac{(2x)^{n-2s}}{(n-2s)!\, s!} = \frac{H_n(x)}{n!}$$

Thus, we have,

$$\boldsymbol{g(x,t) = e^{2xt-t^2} = e^{x^2-(x-t)^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) \dots \dots . (2)}$$

5. RODRIGUE'S FORMULA FOR HERMITE POLYNOMIAL $H_n(x)$

The Rodrigue's formula for Hermite Polynomial $H_n(x)$ is its differential form, which is given by,

$$H_n(x) = (-1)^n . e^{x^2} . \frac{d^n}{dx^n}\left(e^{-x^2}\right) \dots \dots \dots (1)$$

Proof:
We know that the Hermite Polynomial $H_n(x)$ is obtained from the generating function as,

$$g(x,t) = e^{2xt-t^2} = e^{x^2-(x-t)^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) \dots \dots . (2)$$

For all integral values of n and all real values of x, equation (2) can be expressed as,

$$e^{x^2} . e^{-(x-t)^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$$

$$\Rightarrow g(x,t) = e^{x^2}.e^{-(x-t)^2} = \frac{H_0(x)}{0!} + \frac{H_1(x)}{1!}t + \frac{H_2(x)}{2!}t^2 + \cdots + \frac{H_n(x)}{n!}t^n$$

So that

$$\frac{\partial^n}{\partial t^n}[e^{x^2}.e^{-(x-t)^2}]_{t=0} = \frac{H_n(x)}{n!}n! = H_n(x) \ldots\ldots\ldots (3)$$

Now, putting $z = t - x$ at $t = 0$, $z = -x$, so that $\frac{\partial}{\partial t} \equiv \frac{\partial}{\partial z}$

$$\frac{\partial^n}{\partial t^n}[e^{x^2}.e^{-(t-x)^2}]_{t=0} = \frac{\partial^n}{\partial z^n}\left(e^{-z^2}\right) = (-1)^n \frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

$$\therefore \boldsymbol{H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right) \ldots\ldots..(4)}$$

This is required differential form of Hermite Polynomials and is known as Rodrigue's formula.

6. VALUES OF FOR HERMITE POLYNOMIALS

The Rodrigue's formula for Hermite Polynomial $H_n(x)$ is given by,

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right) \ldots\ldots..(1)$$

Putting $n = 0, 1, 2, 3 \ldots\ldots$ in relation (1) we can get the Hermite polynomials of different orders.

$$\therefore H_0(x) = (-1)^0 e^{x^2} \frac{d^0}{dx^0}\left(e^{-x^2}\right) = e^{x^2}.e^{-x^2} = 1$$

$$H_1(x) = (-1)^1 e^{x^2} \frac{d^1}{dx^1}\left(e^{-x^2}\right) = -1.e^{x^2}.-2x.e^{-x^2} = 2x$$

$$H_2(x) = (-1)^2 e^{x^2} \frac{d^2}{dx^2}\left(e^{-x^2}\right) = e^{x^2}.\frac{d}{dx}\left[-2x.e^{-x^2}\right]$$

$$= e^{x^2}.\left(-2x.-2x.e^{-x^2} - 2e^{-x^2}\right) = (4x^2 - 2)$$

Similarly,

$H_3(x) = 8x^3 - 12x$

$H_4(x) = 16x^4 - 48x^2 + 12, etc.$

7. RECURRENCE RELATION FOR HERMITE POLYNOMIALS

The generating function can be used to develop the recurrence relations associated with Hermite Polynomials. Here, we shall derive some of the important recursion/recurrence relations in connection with Hermite Polynomials.

RELATION I: $H'_n(x) = 2nH_{n-1}(x)$

Proof:

We know the generating function for Hermite Polynomial $H_n(x)$ **can be expressed as,**

$$\boldsymbol{g(x,t) = e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!}t^n \ldots\ldots..(1)}$$

Differentiating bothsides with respect to x, we obtain,

$$g'(x,t) = (2t).e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H'_n(x)}{n!} t^n \ldots \ldots . (2)$$

Now using equation (1) on the LHS of the equation (2), we get

$$(2t). \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = \sum_{n=0}^{\infty} \frac{H'_n(x)}{n!} t^n \ldots \ldots . (3)$$

$$2. \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^{n+1} = \sum_{n=0}^{\infty} \frac{H'_n(x)}{n!} t^n \ldots \ldots . (4)$$

Now we shall equate the coefficients of t^n on bothsides of equation (4):

$$2. \frac{H_{n-1}(x)}{(n-1)!} = \frac{H'_n(x)}{n!}$$

$$\therefore H'_n(x) = 2nH_{n-1}(x)$$

RELATION II: $2xH_n(x) = 2nH_{n-1}(x) + H_{n+1}(x)$

Proof:

We have,

$$g(x,t) = e^{2xt-\ ^{2}} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n \ldots \ldots . (1)$$

Differentiating bothsides of equation (1) with respect to t, partially, we get,

$$2(x-t)e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} nt^{n-1}$$

$$\Rightarrow 2(x-t) \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} nt^{n-1}$$

$$\Rightarrow 2x \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n - 2 \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^{n+1} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} nt^{n-1} \ldots . . (2)$$

Now equating the coefficients of t^n on bothsides of equation (2), we get

$$2x. \frac{H_n(x)}{n!} - 2. \frac{H_{n-1}(x)}{(n-1)!} = \frac{H_{n+1}(x)}{(n+1)!}(n+1)$$

$$\Rightarrow 2x. \frac{H_n(x)}{n!} - 2. \frac{nH_{n-1}(x)}{n!} = \frac{H_{n+1}(x)}{(n+1)n!}(n+1)$$

$$\Rightarrow 2xH_n(x) - 2nH_{n-1}(x) = H_{n+1}(x)$$

$$\therefore 2xH_n(x) = 2nH_{n-1}(x) + H_{n+1}(x)$$

RELATION III: $H'_n(x) = 2xH_n(x) - H_{n+1}(x)$

Proof:

From Recurrence Relations I and II, we have

$$H'_n(x) = 2nH_{n-1}(x) \ldots \ldots . (1)$$

$$2xH_n(x) = 2nH_{n-1}(x) + H_{n+1}(x) \ldots \ldots \ldots (2)$$

Substituting the value of $2nH_{n-1}(x)$ in equation (2) from equation (1), we obtain,

$$2xH_n(x) = H'_n(x) + H_{n+1}(x)$$

$$\therefore \boldsymbol{H'_n(x) = 2xH_n(x) - H_{n+1}(x)}$$

RELATION IV: $\boldsymbol{H''_n(x) - 2xH'_n(x) + 2nH_n(x) = 0}$

Proof: We know, $H'_n(x) = 2xH_n(x) - H_{n+1}(x) \ldots \ldots (1)$

Differentiating bothsides with respect to x, we get;

$$H''_n(x) = 2H_n(x) + 2xH'_n(x) - H'_{n+1}(x) \ldots \ldots (2)$$

Again we know,

$$H'_n(x) = 2nH_{n-1}(x) \ldots \ldots (3)$$

Replacing n by $(n+1)$, we get

$$H'_{n+1}(x) = 2(n+1)H_n(x) \ldots \ldots (4)$$

Using equation (4) in equation (2), we get,

$$H''_n(x) = 2H_n(x) + 2xH'_n(x) - 2(n+1)H_n(x)$$

$$\therefore \boldsymbol{H''_n(x) - 2xH'_n(x) + 2nH_n(x) = 0}$$

This is required relation, which also indicates that, $y = H_n(x)$, i.e., the Hermite polynomial is a solution of Hermite's differential equation.

8. ORTHOGONALITY OF HERMITE POLYNOMIALS

A family of functions $f_0(x), f_1(x), f_2(x) \ldots \ldots, f_n(x)$ is said to be orthogonal with respect to a weight $w(x)$ over an interval [a, b] if the following is true:

$$\int_a^b f_m(x)\, f_n(x) w(x) dx = 0 \; for \; m \neq n$$
$$\neq 0 \; for \; m = n$$

Hermite polynomials form an orthogonal set of functions for the weight $w(x) = e^{-x^2}$ over the interval $(-\infty, \infty)$. The exact relation runs as:

$$\int_{-\infty}^{\infty} H_m(x)\, H_n(x) e^{-x^2} dx = 0 \; for \; m \neq n$$
$$= 2^n n! \sqrt{\pi} \; for \; m = n$$

Proof:

We know that $H_m(x)$ is a solution of the Hermite differential equation given by,

$$\boldsymbol{\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2my = 0} \ldots \ldots \ldots (1)$$

So, we can write,

$$H''_m(x) - 2xH'_m(x) + 2mH_m(x) = 0 \ldots \ldots (2a)$$

Similarly,

$$H''_n(x) - 2xH'_n(x) + 2nH_n(x) = 0 \ldots \ldots (2b)$$

Multiplying equation (2a) by $H_n(x)$ and equation (2b) by $H_m(x)$ and subtracting, we get,

$$[H''_m(x)H_n(x) - H''_n(x)H_m(x)] - 2x[H'_m(x)H_n(x) - H'_n(x)H_m(x)] + 2(m-n)H_m(x)H_n(x) = 0 \ldots \ldots (3)$$

$$\Rightarrow \frac{d}{dx}[H'_m(x)H_n(x) - H'_n(x)H_m(x)] - 2x[H'_m(x)H_n(x) - H'_n(x)H_m(x)]$$
$$= 2(n-m)H_m(x)H_n(x) \dots\dots..(4)$$

The above equation is linear differential equation,
So, Integrating factor (I.F.) $= e^{\int -2x dx} = e^{-x^2}$
Multiplying equation (4) by I.F. we get,

$$\frac{d}{dx}[H'_m(x)H_n(x) - H'_n(x)H_m(x)]e^{-x^2} = 2(n-m)e^{-x^2}H_m(x)H_n(x) \dots.(5)$$

Now, integrating bothsides with respect to x, from $x = -\infty$ to $x = \infty$, we obtain:

$$[H'_m(x)H_n(x) - H'_n(x)H_m(x)]e^{-x^2}\Big|_{-\infty}^{\infty} = 2(n-m)\int_{-\infty}^{\infty} e^{-x^2}H_m(x)H_n(x)$$
$$\Rightarrow 0 = 2(n-m)\int_{-\infty}^{\infty} e^{-x^2}H_m(x)H_n(x)$$
$$\therefore \int_{-\infty}^{\infty} e^{-x^2}H_m(x)H_n(x) = 0 \dots\dots..(6a)$$

We also know from the generating function of Hermite polynomial $H_n(x)$ that,

$$g(x,t) = e^{2xt-t^2} = \sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n \dots\dots(7a)$$

Similarly,

$$g(x,s) = e^{2xs-s^2} = \sum_{m=0}^{\infty}\frac{H_m(x)}{m!}s^m \dots\dots(7b)$$

Multiplying (7a) and (7b) we get,

$$e^{2xt-t^2+2xs-s^2} = \sum_{m=0}^{\infty}\frac{H_m(x)}{m!}s^m \sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n$$

$$e^{2xt-t^2+2xs-s^2} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{H_m(x)}{m!}\frac{H_n(x)}{n!}s^m.t^n \dots\dots.(8)$$

Multiplying bothsides of equation (8) by weight $w(x) = e^{-x^2}$ and then integrating from $x = -\infty$ to $x = \infty$, we obtain:

$$\int_{-\infty}^{\infty} e^{-[(x+s+t)^2-2st]}\,dx = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{s^m.t^n}{m!.n!}.\int_{-\infty}^{\infty} H_m(x).H_n(x).e^{-x^2}dx \dots..(9)$$

LHS:

$$\int_{-\infty}^{\infty} e^{-[(x+s+t)^2-2st]}\,dx = e^{2st}\int_{-\infty}^{\infty} e^{-[(x+s+t)^2]}\,dx = e^{2st}\int_{-\infty}^{\infty} e^{-u^2}\,du = e^{2st}.\sqrt{\pi}$$
$$= \sqrt{\pi}.\sum_{m=0}^{\infty}\frac{2^m.s^m.t^m}{m!}$$

Thus from equation (9) using the above result, we get

$$\sqrt{\pi}.\sum_{m=0}^{\infty}\frac{2^m.s^m.t^m}{m!}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{s^m.t^n}{m!.n!}.\int_{-\infty}^{\infty}H_m(x).H_n(x).e^{-x^2}dx\ \ldots\ldots(10)$$

Equating the coefficients of t^n, (if $m=n$) in bothsides of equation (10) we get:

$$\sqrt{\pi}.2^n=\frac{1}{n!}.\int_{-\infty}^{\infty}e^{-x^2}H^2{}_n(x)dx$$

$$\therefore\int_{-\infty}^{\infty}e^{-x^2}H^2{}_n(x)dx=\sqrt{\pi}.2^n.n!\ \ldots\ldots(11a)$$

Combining the results of (6a) and (11a) we can easily write:

$$\int_{-\infty}^{\infty}\boldsymbol{H_m(x)\,H_n(x)e^{-x^2}dx=0\ for\ m\neq n}$$

$$\boldsymbol{=2^n n!\sqrt{\pi}\ for\ m=n}\ldots\ldots\textbf{(12)}$$

6.3.9 INTEGRAL REPRESENTATION OF HERMITE POLYNOMIAL

The integral form of Hermite polynomial is given by,

$$\boldsymbol{H_n(x)=\frac{2^n(-i)^n}{\sqrt{\pi}}e^{x^2}\int_{-\infty}^{\infty}t^n e^{-t^2+2ixt}dt}\ \ldots..\textbf{(1)}$$

(the proof of the equation (1) is given in supplementary exercise)

6.3.10 APPLICATIONS OF HERMITE POLYNOMIALS IN PHYSICS

A. The Linear Harmonic Oscillator Problem in Quantum mechanics

The one dimensional quantum mechanical harmonic oscillator is a state of energy E and is governed by the equation (known as time independent Schrodinger's wave equation)

$$\left[\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)\right]\Psi(x)=E\,\Psi(x)\ \ldots\ldots..(1)$$

Here, $V(x)$ is the potential energy function for the harmonic oscillator, which is given by,

$$V(x)=\tfrac{1}{2}kx^2=\tfrac{1}{2}mw^2x^2\ldots\ldots(2)$$

Inserting equation (2) in equation and then rearranging we obtain,

$$\frac{d^2\Psi}{dx^2}+\frac{2m}{\hbar^2}[E-V(x)]\Psi=0$$

$$\frac{d^2\Psi}{dx^2}+\frac{2m}{\hbar^2}\left[E-\frac{1}{2}mw^2x^2\right]\Psi=0\ \ldots\ldots(3)$$