TDP/BAH1–BSH1/MTMH/14

TDP (Honours) 1st Semester Exam., 2014

MATHEMATICS

(Honours)

FIRST PAPER

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

UNIT—I

Answer any *two* of the following questions : 10×2=20

1. (a) If the sum of any two of the quantities x, y, z be greater than the third, then show that

$$(x+y+z)^3 > 27(y+z-x)(z+x-y)(x+y-z)$$ 3

(b) Find $\phi(2520)$, where ϕ is the Euler phi-function. 3

(c) Deduce the following from the Gregory's series : 4

$$\frac{\pi}{4} = \left(\frac{2}{3}+\frac{1}{7}\right) - \frac{1}{3}\left(\frac{2}{3^3}+\frac{1}{7^3}\right) + \frac{1}{5}\left(\frac{2}{3^5}+\frac{1}{7^5}\right) - \cdots$$

2. *(a)* Prove by mathematical induction that

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \cdots (\cos\theta_n + \sin\theta_n) =$$
$$\cos(\theta_1 + \theta_2 + \cdots + \theta_n) + i\sin(\theta_1 + \theta_2 + \cdots + \theta_n)$$

Hence deduce De Moivre's theorem for positive integral index. 4+1=5

(b) Prove that for any three integers a, b, c $(a, b \neq 0)$, the equation $ax + by = c$ has a solution if and only if (a, b) divides c, where (a, b) denotes g.c.d. of a and b. 5

3. *(a)* Prove that

$$n^n > 1 \cdot 3 \cdot 5 \cdot \cdots (2n-1)$$ 3

(b) Show that $2^{2n+1} - 9n^2 + 3n - 2$ is divisible by 54, for all $n \in N$ (using mathematical induction method). 3

(c) Using Chinese remainder theorem, find the solution of the following system : 4

$$x \equiv 5 \pmod 4$$
$$x \equiv 3 \pmod 7$$
$$x \equiv 2 \pmod 9$$

UNIT—II

Answer any *two* of the following questions : 10×2=20

4. *(a)* If f is a function from Q to Q defined as

$$f(x) = 5x + 7, \text{ for } x \in Q$$

then prove that f is bijective. Also find f^{-1}, where Q is the set of rational numbers. 4

(b) Prove that a non-empty subset H of a group $(G, *)$ is a subgroup of G if and only if for all $a, b \in H$, $a * b^{-1} \in H$. 4

(c) Find the generators of the cyclic group $\{1, -1, i, -i\}$ with respect to the multiplication. 2

5. *(a)* A relation R on the set of integers z is defined in the following way :

$$R = \{(a, b) \in z \times z : a - b \text{ is divisible by } 11\}$$

Show that R is an equivalence relation. 3

(b) Show that the set z of all integers forms a group under the binary operation $*$ defined by

$$a * b = a + b + 1, \ a, b \in z$$ 4

(c) Define even permutation. Show that the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 1 & 2 & 6 & 5 \end{pmatrix}$$

is an even permutation. 3

6. (a) For three sets A, B, C, prove that

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$ 3

(b) Show that the inverse of the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

is an identity permutation. 3

(c) Prove that every subgroup of a cyclic group is cyclic. 4

UNIT—III

Answer any *two* of the following questions : 10×2=20

7. (a) State and prove Lagrange's theorem. 5

(b) Prove that the set of all 2×2 real matrices of the form $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$ forms a field with respect to matrix addition and multiplication. 5

8. *(a)* Prove that the intersection of any two normal subgroups of a group G is a normal subgroup of G. 3

(b) Prove that under matrix addition and multiplication the set of all matrices

$$M = \left\{ \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} : a, b \in R \right\}$$

is a non-commutative ring with divisor of zero. 5

(c) Give an example of a subring. 2

9. *(a)* If f is a homomorphism from a group $(G, \circ)$ to a group $(G', *)$, then prove that $f(e) = e'$, where e and e' are the identity elements of G and G' respectively. 2

(b) Prove that the characteristic of every integral domain is either zero or prime. 3

(c) Prove that every finite integral domain is a field. Give an example of an integral domain which is not a field with proper justification. 5

UNIT—IV

Answer any *two* of the following questions : 10×2=20

10. *(a)* Prove the identity

$$[\vec{a}\times\vec{b},\ \vec{b}\times\vec{c},\ \vec{c}\times\vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]^2$$

If $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar, what is your conclusion about $\vec{a}\times\vec{b}$, $\vec{b}\times\vec{c}$ and $\vec{c}\times\vec{a}$? 5

(b) In any triangle *ABC*, with usual notations, prove that

$$a = b\cos C + c\cos B$$ 5

11. *(a)* A vector $\vec{\gamma}$ is perpendicular to both

$$\vec{\alpha} = 4i + 5j - k,\ \vec{\beta} = i - 4j + 5k$$

and satisfies $\vec{\gamma}\cdot\vec{\delta} = 2$, where $\vec{\delta} = 3i + j - k$, find $\vec{\gamma}$. 5

(b) Find the equation of the plane passing through the point $2i - j - 4k$ and parallel to the plane

$$\vec{r}\,(4i - 12j - 3k) = 7$$

where $\vec{r} = (x, y, z)$. 5

12. *(a)* Find the moment about the point $2i+j-k$ of a force represented by $4i+k$ acting through the point $i-j+2k$. 5

(b) Find the shortest distance between two skew lines $\vec{r}=\vec{r}_1+t\vec{\alpha}$, $\vec{r}=\vec{r}_2+t\vec{\beta}$, where t is a scalar and $\vec{r}_1, \vec{\alpha}, \vec{r}_2, \vec{\beta}$ are the vectors

$\hat{i}-2\hat{j}+3\hat{k}$, $2\hat{i}+\hat{j}+\hat{k}$, $-2\hat{i}+2\hat{j}-\hat{k}$, $-3\hat{i}+\hat{j}+2\hat{k}$ 5

respectively.

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Suvadip Pant

S–1/MTMH/01/15

TDP (Honours) 1st Semester Exam., 2015

MATHEMATICS

(Honours)

FIRST PAPER

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

UNIT—I

Answer any *two* of the following questions : 10×2=20

1. (a) If a, b, c be positive real numbers, then prove that

$$\frac{2}{b+c}+\frac{2}{c+a}+\frac{2}{a+b}>\frac{9}{a+b+c}, \text{ unless } a=b=c$$ 3

(b) If $a=\cos 2\alpha+i\sin 2\alpha$, $b=\cos 2\beta+i\sin 2\beta$, $c=\cos 2\gamma+i\sin 2\gamma$, $d=\cos 2\delta+i\sin 2\delta$, then prove that

$$\sqrt{\frac{ac}{bd}}+\sqrt{\frac{bd}{ac}}=2\cos(\alpha+\gamma-\beta-\delta)$$ 4

(c) Find the least value of $x^{-2}+y^{-2}+z^{-2}$, when $x^2+y^2+z^2=12$. 3

2. (a) State Cauchy-Schward inequality. If x, y, z be positive real numbers such that $x^2+y^2+z^2=27$, then show that

$$x^3+y^3+z^3 \geq 81$$ 4

(b) If $d=\gcd(a, b)$, then prove that

$$\gcd\left(\frac{a}{d}+\frac{b}{d}\right)=1$$ 2

(c) Deduce from the Gregory's series

$$\pi = 2\sqrt{3}\left(1-\frac{1}{3\cdot 3}+\frac{1}{5\cdot 3^2}-\frac{1}{7\cdot 3^3}+\cdots\right)$$ 4

3. (a) State the Chinese remainder theorem. Use it to solve the system

$$x \equiv 1 \pmod 4,\ x \equiv 3 \pmod 7,\ x \equiv 2 \pmod 9$$

1+4=5

(b) Show that $7^{2n}+16n-1$ is divisible by 64 for all $n+N$ (using mathematical induction method). 3

(c) Find $\phi(5186)$, where ϕ is the Euler phi-function. 2

UNIT—II

Answer any *two* of the following questions : 10×2=20

4\. *(a)* Define a group. 2

(b) Prove that the set of rational numbers forms an Abelian group under the operation $*$ defined as follows : 5

$$\text{for any } a, b \in \mathbb{Q}, \quad a * b = \frac{ab}{5}$$

(c) Prove that a relation R defined on a set A is an equivalence relation if and only if R be reflexive and aRb and bRc implies cRa. 3

5\. *(a)* If every element of a group $(G, *)$ is its own inverse, then prove that G is Abelian. 2

(b) Express the following permutation as a product of disjoint cycles and find its order : 2

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 8 & 9 & 6 & 4 & 5 & 2 & 3 & 1 \end{pmatrix}$$

(c) Find the order of the element 8 in the group $(\mathbb{Z}_{14}, +_{14})$. 2

(d) Prove that the order of a cyclic group is the same as the order of its generator and conversely. 4

6\. *(a)* Prove that the union of two subgroups of a group is a subgroup iff one is contained in the other. Give an example to show that the union of two subgroups is not always a subgroup. 4

(b) Show that for any two elements a and b of a group G, a and $b^{-1}ab$ have the same order. 3

(c) In a group G, prove that $(ab)^2 = a^2b^2$ if and only if $(ab)^{-1} = a^{-1}b^{-1}$. 3

UNIT—III

Answer any *two* of the following questions : 10×2=20

7\. *(a)* Let H be a subgroup of a group G and let $a, b \in G$. Prove that $aH = H$ iff $a \in H$. 3

(b) ϕ is a homomorphism from a group G to a group G'. If $g \in G$ be such that $O(g) = n$, then prove that $O(\phi(g))$ divides n. 2

(c) Define normal subgroup. If N is a normal subgroup of a group G and H is any subgroup of G, then prove that NH is a subgroup of G. 1+4=5

8\. *(a)* Prove that the intersection of any collection of ideals of a ring R is an ideal of R. 3

(b) Prove that a non-empty subset S of a ring R is a subring of R iff $a-b \in S$ and $ab \in S$ whenever $a, b \in S$. 4

(c) Prove that the characteristic of an integral domain is either zero or a prime. 3

9. (a) Prove that the set of numbers of the form $a+b\sqrt{3}$, where a and b are rational numbers forms a field under usual addition and multiplication. 5

(b) If in a ring R, $a^2 = a$, for all $a \in R$, then prove that R is commutative. 2

(c) Show that a finite commutative ring without zero divisors has a unity. 3

UNIT—IV

Answer any *two* of the following questions : 10×2=20

10. (a) If $\vec{e}_1$ and $\vec{e}_2$ be two unit vectors and θ be the angle between them, then show that

$$2\sin\frac{\theta}{2} = |\vec{e}_1 - \vec{e}_2|$$ 5

(b) If $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ are the three vectors, then prove that

$$\vec{\alpha} \times (\vec{\beta} \times \vec{\gamma}) = (\vec{\alpha} \cdot \vec{\gamma})\vec{\beta} - (\vec{\alpha} \cdot \vec{\beta})\vec{\gamma}$$ 5

11. *(a)* In any triangle ABC, with usual notations, prove that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$ 4

(b) Find the work done by a particle acted on by a force $5\hat{i} + 10\hat{j} + 15\hat{k}$ for the displacement from the point $\hat{i} + 3\hat{k}$ to $3\hat{i} - \hat{j} - 6\hat{k}$. 3

(c) Show that the moment of a force $4\hat{i} + 2\hat{j} + \hat{k}$ through the point $5\hat{i} + 2\hat{j} + 4\hat{k}$ about the point $3\hat{i} - \hat{j} + 3\hat{k}$ is $\hat{i} + 2\hat{j} - 8\hat{k}$. 3

12. *(a)* If the volume of a tetrahedron is 3 cu. units and three of its vertices have position vectors $(1, 1, 0)$, $(1, 0, 1)$ and $(2, -1, 1)$, then find the locus of the fourth vertex. 4

(b) Find the equation of the plane passing through the points $(2, -1, 4)$, $(3, 4, 7)$ and $(-2, 3, -1)$. 3

(c) If $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$, then prove that the vectors $\vec{a}$ and $\vec{b}$ are orthogonal. 3

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S–1/MTMH/01/17

TDP (Honours) 1st Semester Exam., 2017

MATHEMATICS

(Honours)

FIRST PAPER

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

UNIT—I

Answer any *two* of the following questions : 10×2=20

1. *(a)* If a, b, c be positive and $a+b+c=1$, then show that $\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right)\geq 8$ and the least value of $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ is 9. 2+2=4

(b) State De Moivre's theorem and establish it for positive integral value of n. 1+2=3

(c) Find the g.c.d. of 256 and 1166 and hence express the g.c.d. as a linear combination of 256 and 1166. 2+1=3

2. *(a)* If each of a, b, c, d be greater than 1, then show that

$$8(abcd+1) > (a+1)(b+1)(c+1)(d+1)$$ 3

(b) Using the principle of finite induction, prove that $1+2+3+\ldots+n = \frac{n(n+1)}{2}$, for all integers $n \geq 1$. 3

(c) Prove that the equation $ax+by=c$ has a solution iff (a, b) divides c ($a \neq 0$, $b \neq 0$ and c are integers). 4

3. *(a)* Use Chinese remainder theorem to solve the simultaneous systems of linear congruences : 4

$$x \equiv 2 \pmod 3$$
$$x \equiv 3 \pmod 5$$
$$x \equiv 2 \pmod 7$$

(b) Deduce from the Gregory's series

$$\tan^{-1}\frac{1-\cos\theta}{1+\cos\theta} = \tan^2\frac{\theta}{2} - \frac{1}{3}\tan^6\frac{\theta}{2} + \frac{1}{5}\tan^{10}\frac{\theta}{2} - \ldots,$$

if $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ 3

(c) Prove that

$$\sin\left(i\log\frac{a-ib}{a+ib}\right) = \frac{2ab}{a^2+b^2}$$ 3

UNIT—II

Answer any *two* of the following questions : 10×2=20

4. (a) Show that the set Z of all integers can be partitioned into equivalence classes by a relation ρ, defined on it by $x \rho y \Leftrightarrow x^2 - y^2$ is divisible by 5. Find the distinct equivalence classes of Z by ρ. 3+2=5

(b) Show that identity element in a group $(G, *)$ is unique. 2

(c) Find a subgroup of $G = \{1, -1, i, -i\}$, where $(G, \bullet)$ forms a group under multiplication '$\bullet$'. 3

5. (a) Determine whether the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 5 & 6 & 3 & 7 & 4 & 2 & 1 \end{pmatrix}$$

is even or odd. 2

(b) Show that the set

$$M_2(R) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

is an Abelian group under matrix multiplication. Is it a cyclic group? 3+1=4

(c) Let $f : R \to R$ and $g : R \to R$ be two mappings defined by $f(x) = x^2$ and $g(x) = \cos x$ respectively. Find $f \circ g$ and $g \circ f$. Is the composition '$\circ$' commutative? 1½+1½+1=4

6. *(a)* If $X = \{x : x \in \mathbb{R}, x \neq 0\}$, then prove that the mapping $f : X \to X$ defined by $f(x) = \dfrac{1}{x}$ is one-one and onto, where $\mathbb{R}$ is the set of real numbers. 3

(b) Prove that the union of two subgroups is a subgroup iff one is contained in the other. 3

(c) Define cyclic group. Show that the set $\{1, \omega, \omega^2\}$ forms a multiplicative cyclic group (where ω is the cube root of unity) and hence find the generators. 4

UNIT—III

nswer any *two* of the following questions : 10×2=20

7. *(a)* Prove that any two right (or left) cosets of a subgroup H of G are either disjoint or identical. 4

(b) Let $(G, \cdot) = (Z, +)$ and $(G', *) = mz : z(Z, +)$ and $f : G \to G'$ be defined by $f(z) = mz \ \forall \ z \in Z$, the set of integers. Examine whether f is isomorphism or not. 4

(c) Prove that identity element of a ring is unique. 2

8. (a) Prove that the set of integers forms a commutative ring with unity for ordinary addition and multiplication. 4

(b) Let S and T be two subrings of a ring R. Show that $S \cap T$ is a subring of R. 3

(c) Let $G = S_3$, $G' = (\{1, -1\}, \cdot)$ and $\phi : G \to G'$ is defined by

$$\phi(\alpha) = 1 \text{ if } \alpha \text{ is an even permutation in } S_3$$
$$= -1 \text{ if } \alpha \text{ is an odd permutation in } S_3$$

Determine $\ker \phi$. 3

9. (a) Let $\phi : (G, \circ) \to (G', *)$ be an isomorphism. Prove that $\phi^{-1} : (G', *) \to (G, \circ)$ is also an isomorphism. 3

(b) Define normal subgroup. Prove that the centre of a group G is a normal subgroup of G. 1+3=4

(c) Prove that every field is an integral domain. 3

UNIT—IV

Answer any *two* of the following questions : 10×2=20

10. (a) Verify whether the following points are collinear or not $-2\vec{a} + 3\vec{b} + 5\vec{c}$, $\vec{a} + 2\vec{b} + 3\vec{c}$, $7\vec{a} - \vec{c}$. 3

(b) On the line $\frac{x-3}{9}=\frac{y+4}{6}=\frac{z+2}{2}$, find two points each of whose distance from $(3, -4, -2)$ is 22. 4

(c) Find the moment about $(1, -1, 1)$ of the force $3\hat{i}+4\hat{j}-5\hat{k}$ acting at $(1, 0, -2)$. 3

11. (a) Show that the torque about the point $3\hat{i}-\hat{j}+3\hat{k}$ of a force represented by $4\hat{i}+2\hat{j}+\hat{k}$ passing through the point $5\hat{i}+2\hat{j}+4\hat{k}$ is $\hat{i}+2\hat{j}-8\hat{k}$. 3

(b) Show that

$$\left[\vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}\ \ \vec{a}\times\vec{b}\right]=[\vec{a}\ \ \vec{b}\ \ \vec{c}]^2$$

3

(c) Find a unit vector, in the plane of the vectors $(\hat{i}+2\hat{j}-\hat{k})$ and $(\hat{i}+\hat{j}-2\hat{k})$, which is perpendicular to the vector $(2\hat{i}-\hat{j}+\hat{k})$. 4

12. (a) Show that the volume of the parallelopiped whose edges are represented by $(3\hat{i}+2\hat{j}-4\hat{k})$, $(3\hat{i}+\hat{j}+3\hat{k})$ and $(\hat{i}-2\hat{j}+\hat{k})$ is 49 cubic units. 4

(b) A particle being acted on by constant forces $(4\hat{i}+\hat{j}-3\hat{k})$ and $(3\hat{i}+\hat{j}-\hat{k})$ is displaced from the point $(\hat{i}+2\hat{j}+3\hat{k})$ to the point $(5\hat{i}+4\hat{j}-\hat{k})$. Find the total work done by the forces. 3

(c) If $\vec{a}$, $\vec{b}$, $\vec{c}$ be three unit vectors such that $\vec{a} \times (\vec{b} \times \vec{c}) = \frac{1}{2}\vec{b}$, then find the angles which $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$, $\vec{b}$ and $\vec{c}$ being non-parallel. 3

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S–1/MTMH/01/16

TDP (Honours) 1st Semester Exam., 2016

MATHEMATICS

(Honours)

FIRST PAPER

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

UNIT—I

Answer any *two* of the following questions : 10×2=20

1. *(a)* If a and b be two unequal positive real numbers, then prove that

$$\sqrt{ab} > \frac{2}{\frac{1}{a}+\frac{1}{b}}$$ 3

(b) If

$$\cos\alpha+\cos\beta+\cos\gamma=0=\sin\alpha+\sin\beta+\sin\gamma$$

then prove that

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=\sin^2\alpha+\sin^2\beta+\sin^2\gamma=\frac{3}{2}$$ 4

(c) Using Gregory's series, prove that

$$\frac{\pi}{8} = \frac{1}{1\cdot 3} + \frac{1}{5\cdot 7} + \frac{1}{9\cdot 11} + \cdots$$ 3

2. (a) Find the g.c.d. and l.c.m. of 306, 657. 3

(b) Prove that

$$\tan^{-1}\left[\frac{i(x-a)}{x+a}\right] = -\frac{i}{2}\log\frac{a}{x}$$ 4

(c) Prove that if p is a prime and $p|ab$, then $p|a$ or $p|b$. 3

3. (a) By using the principle of induction, prove that

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{4n^3 - n}{3}$$

for each $n \geq 1$. 3

(b) Prove that for two integers a and b with $b > 0$, there exist unique integers q and r such that $a = bq + r,\ 0 \leq r < b$. 4

(c) Find the general solution of

$$170x - 455y = 625$$ 3

UNIT—II

Answer any *two* of the following questions : 10×2=20

4. *(a)* Show that whether the relation $R = \{(x, y) \mid x \geq y\}$ is reflexive, symmetric or transitive in the set of real numbers. 3½

(b) Let $(G, 0)$ be a finite cyclic group of order $n > 1$, generated by a. Prove that for a positive integer r, a^r is also a generator of the group iff r is less than n and prime to n. 4

(c) If a be an element of a group G such that $a^2 = a$, then show that $a = e$. 2½

5. *(a)* If $f : Q \to Q$ is defined by $f(x) = ax + b$, where $a, b, x \in Q$, the set of rationals, then show that whether f is one-one and onto. 3

(b) Show that the set of all real numbers is a groupoid but not a semigroup under the operation $\circ$ defined by $a \circ b = a + 3b$, $\forall\ a, b \in R$. 2

(c) Prove that of the $n!$ permutations on n symbols $(n > 1)$, $\frac{n!}{2}$ are even permutations and $\frac{n!}{2}$ are odd permutations. 5

6. *(a)* Examine if the relation

$$\rho = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : 3a + 4b \text{ is divisible by } 7\}$$

on the set $\mathbb{Z}$ is an equivalant relation. 4

(b) Find the order of the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 1 & 3 & 2 & 8 & 6 & 7 \end{pmatrix}$$ 2

(c) Show that every subgroup of a cyclic group is cyclic. 4

UNIT—III

Answer any *two* of the following questions : 10×2=20

7. *(a)* Let G be the additive group of integers and H the additive subgroups of even integers with zero. Find the cosets of H in G. 2

(b) Prove that every finite integral domain is a field. 4

(c) Define characteristic of a ring. Prove that in a ring R for all $a \in R$, $a \cdot 0 = 0 \cdot a = 0$. 1+3=4

8. *(a)* Define simple group. Whether every group of prime order is simple. Justify your answer. 1+2=3

(b) Define homomorphism of two groups. Let

$$G = (\mathbb{Z}, +),\ G' = (2\mathbb{Z}, +)$$

and a mapping $\phi : G \to G'$ be defined by $\phi(a) = 2a,\ a \in G$. Examine if ϕ is a homomorphism. 1+2=3

(c) State and prove Lagrange's theorem. 4

9. *(a)* If H be a subgroup of a group G and $[G:H] = 2$, then prove that H is normal in G. 3

(b) If $\phi : G \to G'$ be a homomorphism, then prove that $\phi(G)$ is a subgroup of G', where $\phi(G)$ is the homomorphic image of ϕ. 3

(c) Prove that $(\mathbb{Z}_6, +, \circ)$ is a commutative ring with unity having zero divisor. 4

UNIT—IV

Answer any *two* of the following questions : 10×2=20

10. *(a)* Find the unit vector perpendicular to each of $\vec{a} = 6\hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - 6\hat{j} - 2\hat{k}$. 3

(b) Prove the following : 2+2=4

(i) $[\vec{a} + \vec{b}\ \ \vec{b} + \vec{c}\ \ \vec{c} + \vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}]$

(ii) $(\vec{b} \times \vec{c}) \times (\vec{b} \times \vec{a}) = [\vec{a}\ \vec{b}\ \vec{c}]\,\vec{b}$

(c) Find the equation of the plane passing through the points (−1, 1, 2), (1, −2, 1) and (2, 2, 4). 3

11. (a) Prove by vector method, the trigonometrical formula

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$ 3

(b) Show that the four points whose position vectors are given as

$$-6\vec{a} + 3\vec{b} + 2\vec{c},\ 3\vec{a} - 2\vec{b} + 4\vec{c},\ 5\vec{a} + 7\vec{b} + 3\vec{c}$$
$$\text{and } -13\vec{a} + 17\vec{b} - \vec{c}$$

are coplanar; *a*, *b*, *c* being three non-coplanar vectors. 4

(c) Find the vector equation of the st[illegible] line passing through the points $(\hat{i} + \hat{j} + \hat{k})$ and $(3\hat{i} + 2\hat{j} - \hat{k})$. 3

12. (a) A particle acted on by constant forces $5\hat{i} + 2\hat{j} + \hat{k}$ and $2\hat{i} - \hat{j} - 3\hat{k}$ is displaced from the origin to the point $4\hat{i} + \hat{j} - 3\hat{k}$. Find the total work done by the forces. 3

(b) Show that the volume of a tetrahedron whose vertices are $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ is

$$\frac{1}{6}[\vec{a}-\vec{d}\ \ \vec{b}-\vec{d}\ \ \vec{c}-\vec{d}] \qquad 3$$

(c) $P(1, 3, -1)$, $Q(0, 1, 6)$, $R(-1, 3, 1)$ are three points in space. Find the coordinates of a point S on the y-axis such that the volume of the tetrahedron $PQRS$ is 10. 4

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S–1/MTMH/01/18

TDP (Honours) 1st Semester Exam., 2018

MATHEMATICS

(Honours)

FIRST PAPER

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **eight** questions, taking **two** from each Unit

UNIT—I

1. *(a)* If a, b, c be any three positive real numbers, then prove that

$$\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}+\frac{a^2+b^2}{a+b} \geq a+b+c$$ 3

(b) If $\log\sin(\theta+i\phi)=\alpha+i\beta$, then prove that $2e^{2\alpha}=\cosh 2\phi-\cos 2\theta$; θ, ϕ, α, β are real. 3

(c) Use Chinese remainder theorem to solve the system of linear congruences : 4

$$x \equiv 1 \pmod 4$$
$$x \equiv 3 \pmod 7$$
$$x \equiv 2 \pmod 9$$

2. *(a)* State mth power theorem. If a, b, c be three positive numbers such that their sum is unity, then find the least value of

$$\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2+\left(c+\frac{1}{c}\right)^2$$

applying mth power theorem. 1+3=4

(b) What is the remainder when the following sum is divided by 4? 3

$$1^5+2^5+3^5+4^5+\ldots+100^5$$

(c) Find the general values and the principal value of $(-1+i)^i$. 3

3. *(a)* Find $\phi(2520)$, where ϕ denotes Euler phi-function. 3

(b) Show that the roots of $x^7=1$ are multiples of α, where $\alpha=\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}$ and also prove that the roots of $x^2+x+2=0$ are $\alpha+\alpha^2+\alpha^4$ and $\alpha^3+\alpha^5+\alpha^6$. 3

(c) Write the second principle of mathematical induction. By using the principle of induction, prove that

$$1^2+3^2+5^2+\ldots+(2n-1)^2=\frac{4n^3-n}{3}$$

for each $n\geq 1$. 1+3=4

UNIT—II

4. *(a)* Let $f : S \to \mathbb{R}$ be defined by $f(x) = [x] - x$, $x \in S$ and $g : S \to \mathbb{R}$ be defined by $g(x) = 1 - x$, where $S = \{x \in \mathbb{R} : 1 \le x < 2\}$. Show that $f = g$. 3

(b) Show that the set z of all integers forms a group under binary operation $*$ defined by $a * b = a + b + 1$; $a, b \in z$. 4

(c) Determine whether the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 5 & 6 & 3 & 1 \end{pmatrix}$$

is even or odd. 3

5. *(a)* Let z be the set of all integers. If $a, b \in z$, we define $a \equiv b \pmod 5$, if $(a - b)$ is divisible by 5. Prove that '$\equiv$' is an equivalence relation on z. 3

(b) Prove that the intersection of two subgroups of a group is again a subgroup of the group. Give an example to show that the union of two subgroups is not always a subgroup. 3+1=4

(c) Prove that every subgroup of a cyclic group is cyclic. 3

6\. (a) Let $S = \{x : x \text{ is a real number}\} - \{-1\}$. Define $*$ on S by $a * b = a + b + a \cdot b$. Show that $\langle S, *\rangle$ forms a group. 3

(b) Prove that the necessary and sufficient condition for a non-empty subset S of a group $(G, *)$ to be a subgroup is

$$a \in S,\ b \in S \Rightarrow a * b^{-1} \in S$$

where b^{-1} is the inverse of b in G. 4

(c) Show that $(\mathbb{Z}_4, +)$ is a cyclic group. 3

UNIT—III

7\. (a) Prove that any two left cosets of a subgroup are either disjoint or identical. 3

(b) Let $\phi : (G, \circ) \to (G', *)$ be a homomorphism. Prove that $\ker \phi$ is a normal subgroup of G. 3

(c) Show that the set of even integers is a subring of the ring of integers under usual addition and multiplication. 4

8\. (a) Prove that the set $S = \{a + b\sqrt{5} : a, b \in Q\}$ is a subfield of the field $\mathbb{R}$. 3

(b) Prove that a finite integral domain is a field. 3

(c) Prove that the characteristic of an integral domain is either zero or a prime. 4

9. (a) Are the groups $(\mathbb{Z}, +)$ and $(Q, +)$ isomorphic? Justify. 3

(b) Show that the set of all integers forms an integral domain but not a field. 4

(c) Prove that a subset S of a field $(F, +, \cdot)$ having at least two elements is a subfield iff for any two elements $x, y (y \neq 0)$ of S, $x - y$ and $x \cdot y^{-1}$ belong to S. 3

UNIT—IV

10. (a) If two medians of a triangle are equal, then by vector method prove that the triangle is isosceles. 3

(b) Find the torque about the point $(1, 2, -1)$ of a force represented by $3\hat{i} + \hat{j} + \hat{k}$ acting through the point $(2, -1, 3)$. 3

(c) Find the shortest distance between two skew lines $\vec{r} = \vec{r_1} + t\vec{\alpha}$ and $\vec{r} = \vec{r_2} + t\vec{\beta}$ where $\vec{r_1} = (-5, -5, 1)$, $\vec{\alpha} = (3, 2, -2)$, $\vec{r_2} = (9, 0, 2)$, $\vec{\beta} = (6, -2, -1)$. 4

11. *(a)* Reduce the expression

$$[(\vec{b}+\vec{c}),\ (\vec{c}+\vec{a}),\ (\vec{a}+\vec{b})]$$

in its simplest form. Hence prove that it vanishes when $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar.

3+1=4

(b) A particle acted on by constant forces $5\hat{i}+2\hat{j}+\hat{k}$ and $2\hat{i}-\hat{j}-3\hat{k}$ is displaced from the origin to the point $4\hat{i}+\hat{j}-3\hat{k}$. Find the total work done by the forces. 3

(c) Find the vector equation of the plane passing through the points $2\hat{i}-3\hat{j}-\hat{k}$, $5\hat{i}-7\hat{j}+9\hat{k}$ and $2\hat{i}-\hat{j}+3\hat{k}$. 3

12. *(a)* If $\vec{\alpha}=\hat{i}+\hat{j}-2\hat{k}$, $\vec{\beta}=-\hat{i}+2\hat{j}+3\hat{k}$, $\vec{\gamma}=5\hat{i}+8\hat{k}$, then determine scatters c and d such that $\vec{\gamma}-c\vec{\alpha}-d\vec{\beta}$ is perpendicular to both $\vec{\alpha}$ and $\vec{\beta}$. 3

(b) Prove by vector method that in a triangle ABC

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

where $BC=a$, $CA=b$, $AB=c$. 3

(c) Show that the line through $P(4, -3, -1)$ and parallel to the vector $(1, 4, 7)$ is

$$\frac{x-4}{1} = \frac{y+3}{4} = \frac{x+1}{7}$$

and find two points on it at a distance of $\sqrt{1056}$ from P. 4

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[redacted]

S–1/MTMH/01/19

TDP (Honours) 1st Semester Exam., 2019

MATHEMATICS

(Honours)

FIRST PAPER

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **eight** questions, taking **two** from each Unit

UNIT—I

1. *(a)* If n be a positive integer, then prove that

$$\left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n = \cos\left(\frac{n\pi}{2}-n\theta\right)+i\sin\left(\frac{n\pi}{2}-n\theta\right)$$ 4

(b) If $a \mid bc$ and $(a, b)=1$, then show that $a \mid c$. 2

(c) If a, b, c be three positive numbers and $a+b+c=1$, then show that the least value of $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ is 9 and prove that

$$(1-a)(1-b)(1-c) > 8abc$$ 4

2. *(a)* Prove that

$$\frac{1}{2\sqrt{n+1}} < \frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \dots, \frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}$$ 4

(b) Find the values of $\phi(1025)$, $\phi(1125)$ and $\phi(1024)$, where ϕ is the Euler's ϕ function. 3

(c) Using Gregory's series, prove that

$$\frac{\pi}{8} = \frac{1}{1\cdot 3} + \frac{1}{5\cdot 7} + \frac{1}{9\cdot 11} + \cdots$$ 3

3. *(a)* If a, b, c, d be four positive numbers, then show that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \geq 4$$ 3

(b) Use Chinese remainder theorem to solve the simultaneous systems of linear congruences

$$x \equiv 3 \pmod{6}$$
$$x \equiv 5 \pmod{7}$$
$$x \equiv 2 \pmod{11}$$ 4

(c) Show that the product of all the values of $(1+i\sqrt{3})^{\frac{3}{4}}$ is 8. 3

UNIT—II

4. *(a)* Show that the set of cube roots of unity is a finite Abelian group with respect to multiplication. 3

(b) Define order of an element in a group. Prove that the orders of the elements a and $x^{-1}ax$ are the same, where a and x are two elements of the group. 1+4=5

(c) Verify whether

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 3 & 1 & 2 & 6 & 7 & 5 \end{pmatrix}$$

is an odd permutation or an even permutation. 2

5. *(a)* Define dihedral group. State and prove the necessary and sufficient conditions for a non-empty subset of a group to be a subgroup. 1+1+4=6

(b) Show that the mapping $f : R \to (-1, 1)$ defined by $f(x) = \dfrac{x}{1+x^2}$, $x \in R$ is not bijective, where R is the set of real numbers. 4

6. *(a)* Give an example to show that the union of two subgroups may not be a subgroup. 2

(b) A relation R is defined on the set Z (the set of all integers) by ${}_aR_b$ if and only if $2a+3b$ be divisible by 5, for all $a, b \in Z$. Prove that R is an equivalence relation. 4

(c) Define cyclic group. Prove that the order of the cyclic group is the same as the order of its generator. 1+3=4

UNIT—III

7. (a) Prove that every finite integral domain is a field. Show that the set of numbers of the form $a+b\sqrt{2}$, where a and b are integers, does not form an integral domain under ordinary addition and multiplication. 4+2=6

(b) Define normal subgroup. Show that N is a normal subgroup of a group G if and only if $gNg^{-1} = N$, for every $g \in G$. 4

8. (a) Prove that the characteristic of an integral domain is either zero or a prime number. 4

(b) Prove that the homomorphism ϕ of the ring R into the ring R' is an isomorphism if and only if the kernel of ϕ is $\{0\}$. 3

(c) Show that if the two right cosets Ha and Hb be distinct, then the two left cosets $a^{-1}H$ and $b^{-1}H$ are distinct. 3

9. (a) Prove that the set of all 2×2 real matrices of the form $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$ forms a field with respect to matrix addition and multiplication. 4

(b) If U is an ideal of the ring R, then prove that R/U is a ring and is a homomorphic image of R. 4+2=6

UNIT—IV

10. (a) Prove by vector method, the formula

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$ 4

(b) Find the value of the constant d, such that the vectors $(2\hat{i}-\hat{j}+\hat{k})$, $(\hat{i}+2\hat{j}-3\hat{k})$ and $(3\hat{i}+d\hat{j}+5\hat{k})$ are coplanar. 4

(c) If $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}$, then can you conclude that $\vec{b}=\vec{c}$? Give reasons for your answer. 2

11. *(a)* For three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ prove that

$$[\vec{b}\times\vec{c} \;\; \vec{b}\times\vec{c} \;\; \vec{b}\times\vec{c}] = [\vec{a} \;\; \vec{b} \;\; \vec{c}]^2$$ 3

(b) Find a unit vector which is perpendicular to each of the vectors $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} + \hat{k}$. Show, by vector method, that the straight line joining the mid-points of the two non-parallel sides of a trapezium is half the sum of the two parallel sides. 1+4=5

(c) Find the vector equation of the plane passing through the three points $(-1, 1, 2)$, $(1, -2, 1)$ and $(2, 2, 4)$. 2

12. *(a)* A particle being acted on by constant forces $(4\hat{i} + \hat{j} - 3\hat{k})$ and $(3\hat{i} + \hat{j} - \hat{k})$ is displaced from the point $(\hat{i} + 2\hat{j} + 3\hat{k})$ to the point $(5\hat{i} + 4\hat{j} - \hat{k})$. Find the total work done by the forces. 4

(b) Show that the vector $\vec{b} - \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a}$ is perpendicular to the vector $\vec{a}$. 2

(c) Find by vector method, the volume of the tetrahedron, the coordinates of whose vertices are (0, 1, 2), (3, 0, 1), (1, 1, 1) and (4, 3, 2). 4

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TDP (Honours) 1st Semester Exam., 2021
(Held in 2022)

MATHEMATICS

(Honours)

FIRST PAPER

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **eight** questions, taking **two** from each Unit

The symbols used have their usual meanings

UNIT—I

1. (a) Separate $\sin(x+iy)$ into real and imaginary parts, x, y being real. Also show that

$$|\sin(x+iy)|^2 = \sin^2 x + \frac{1}{4}(e^y - e^{-y})^2$$ 5

(b) If p, q, r, s be all positive, then show that

$$\frac{(p+q)rs}{ps+qr} \leq \frac{pr+qs}{p+q}$$ 3

(c) Find the values of $(1+i\sqrt{3})^{\frac{1}{2}}$. 2

2. (a) If $x+\frac{1}{x}=2\cos\frac{\pi}{7}$, then show that

$$x^7+\frac{1}{x^7}=-2$$ 3

(b) Deduce the following using the Gregory's series :

$$\pi = 2\sqrt{3}\left[1-\frac{1}{3\cdot 3}+\frac{1}{5\cdot 3^2}-\frac{1}{7\cdot 3^3}+\cdots\right]$$ 4

(c) Find GCD of 723 and 24 by Euclidian algorithm. 3

3. (a) Show that the product of all the values of $(1+i\sqrt{3})^{3/4}$ is 8. 4

(b) Prove that if $ac \equiv bc (\text{mod } m)$ and GCD $(c, m)=1$, then $a \equiv b (\text{mod } m)$. 2

(Continued)

(c) Use Chinese remainder theorem to solve the simultaneous system of linear congruences

$$x \equiv 3 \pmod 6$$
$$x \equiv 5 \pmod 7$$
$$x \equiv 2 \pmod{11}$$ 4

UNIT—II

4. (a) If R be an equivalence relation in a set A, then show that R^{-1} is also an equivalence relation. 3

(b) Let $G = S_3$, $G' = (\{1, -1\}, *)$ and $\phi : G \to G'$ is defined by

$\phi(\alpha) = 1$ if α is an even permutation in S_3

$= -1$ if α is an odd permutation in S_3

Determine $\ker \phi$. 4

(c) Show that $(A-B)$ and $A \cap B$ are disjoint set. 3

5. (a) Define cyclic group. Prove that every subgroup of a cyclic group is cyclic. 1+3=4

(b) Define permutation group. If $a = (1\ 2\ 3\ 4)$, then show that the set $\{a, a^2, a^3, a^4\}$ forms a cyclic group. 4

(c) Define order of a group and order of the element of the group with example. 2

6. (a) Prove that the intersection of any two subgroups of a group $(G, *)$ is again a subgroup of $(G, *)$. Is their union a subgroup? Justify your answer. 3+1=4

(b) Prove that if a is a generator of a cyclic group, then a^{-1} is also a generator. 4

(c) Define alternating group. Give an example of alternating group. 1+1=2

UNIT—III

7. (a) If R be a ring such that $a^2 = a, \forall a \in R$, then prove that—

(i) $ab = ba, \forall a, b \in R$

(ii) $a + a = 0, \forall a \in R$ 2+2=4

(b) Define integral domain. Show that every field is an integral domain. Is the converse of the theorem is true? Justify your answer. 1+4+1=6

8. (a) Define normal subgroup. If N is a normal subgroup of a group G and H is any subgroup of G, then prove that NH is a subgroup of G. 4

(b) Prove that in a group G, the subset $A = \{a \in G : ag = ga, \forall g \in G\}$ is a normal subgroup of G. 3

(c) Verify whether the set of real numbers of the form $b\sqrt{2}$, with b rational, forms a ring or not. 3

9. (a) Show that the modulo 5 system for the set $\{0, 1, 2, 3, 4\}$ is a field with respect to addition and multiplication under modulo system. 3

(b) Are the groups $(Z, +)$ and $(Q, +)$ isomorphic? Justify. 3

(c) Let $\phi : (G, \circ) \rightarrow (G', *)$ be a homomorphism. Prove that $\ker \phi$ is a normal subgroup of G. 4

UNIT—IV

10. (a) In any triangle ABC, with usual notations, prove that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$ 4

60
72X
720

(b) Find the area of the parallelogram whose diagonals are $3\hat{i}+\hat{j}-2\hat{k}$ and $\hat{i}-3\hat{j}+4\hat{k}$. 3

(c) Find the vectorial equation of the plane, passing through three points whose position vectors are (–4, 1, 1), (2, 1, –2) and (8, –3, –2). 3

1320
693

11. (a) Find the torque about the point $B(3, -1, 3)$ of a force $F(4, 2, 1)$ passing through the point $A(5, 2, 4)$. 3

1329
693
17

(b) If $\vec{\alpha}, \vec{\beta}, \vec{\gamma}, \vec{\delta}$ are vectors such that $\vec{\alpha}\times\vec{\beta}=\vec{\gamma}\times\vec{\delta}$ and $\vec{\alpha}\times\vec{\gamma}=\vec{\beta}\times\vec{\delta}$, then show that the vectors $\vec{\alpha}-\vec{\delta}$ and $\vec{\beta}-\vec{\gamma}$ are collinear. 3

(c) Prove that the four points $2\hat{i}+3\hat{j}-\hat{k}$, $\hat{i}-2\hat{j}+3\hat{k}$, $3\hat{i}+4\hat{j}-2\hat{k}$ and $\hat{i}-6\hat{j}+6\hat{k}$ are coplanar. 4

12. (a) Show, by vector method, that the angle in a semicircle is a right angle. 4

(b) Determine the values of λ and μ for which the vectors $-3\hat{i}+4\hat{j}+\lambda\hat{k}$ and $\mu\hat{i}+8\hat{j}+6\hat{k}$ are collinear. 1+1=2 *(c)* Find by vector method, the volume of the tetrahedron, whose coordinate vectors are (0, 1, 2), (3, 0, 1), (1, 1, 1) and (4, 3, 2). 4