23/02/2022

TDP (Honours) 5th Semester Exam., 2021
(Held in 2022)

MATHEMATICS

(Honours)

FIFTH PAPER

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **eight** questions, taking **two** from each Unit

UNIT—I

1. *(a)* Discuss the convergence of the sequence $\{x_n\}$ defined by

$$x_n = \left(1+\frac{1}{n}\right)^n$$

for $n \in N$. 4

(b) State nested intervals theorem. Justify with an example that the condition of closedness of the intervals in the statement cannot be relaxed. 1+2=3

(c) Find the set of limit points of the set

$$\left\{\frac{1}{n} : n \in N\right\}$$ 3

2. *(a)* Show that a sequence of real numbers is convergent iff it is Cauchy. 5

(b) Examine the sequence $\{x_n\}$, defined by

$$x_1 = \sqrt{2},\ x_{n+1} = \sqrt{2x_n},\ n \in N$$

for converges. If yes, then find its limit. 5

3. *(a)* Show that every convergent sequence is bounded. Is the converse true? Justify your answer. 3

(b) Find a countable and an uncountable covering of the set R of real numbers. 2

(c) Show that Q, the set of irrational numbers is Archimedean ordered field but not ordered complete. 5

UNIT—II

4. *(a)* Discuss the Riemann integrability of the function $f:[0, 1] \to R$ given by

$$f(x) = \begin{cases} \frac{1}{q}, & \text{when } x = \frac{p}{q},\ p, q \in N, (p, q) = 1 \\ 0, & \text{when } x \text{ is irrational or zero} \end{cases}$$ 4

2021

(b) Let $f : [a, b] \to R$ be continuous except for finitely many points in $[a, b]$. Show that f is Riemann integrable on $[a, b]$. 4

(c) Show that the sum of two Riemann integrable functions is Riemann integrable. 2

5. *(a)* Let f be a bounded function on $[a, b]$ and P be a partition of $[a, b]$. If P' is a refinement of P, then show that

$$L(P, f) \le L(P', f) \le U(P', f) \le U(P, f)$$ 4

(b) State Bonnet's form of second mean value theorem of integral calculus. Applying this theorem, show that

$$\left| \int_a^b \frac{\sin x}{x} dx \right| \le \frac{2}{a}$$

where $0 < a < b < \infty$. 2+3=5

(c) State first mean value theorem for integrals. 1

6. *(a)* A function $f : [0, 1] \to R$ is defined by

$$f(x) = \frac{1}{a^{r-2}}$$

where $\frac{1}{a^r} < x \le \frac{1}{a^{r-1}}$, $r = 1, 2, 3, \cdots$ and $a > 2$. Is f Riemann integrable on $[0, 1]$? If so, then evaluate

$$\int_0^1 f(x)\, dx$$ 5

(Turn Over)

(b) Prove with the help of an example that the equation

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

is not always valid. 2

(c) Show that second mean value theorem does not hold good in $[-1, 1]$ for the functions $f(x) = g(x) = x^2$. Also test the validity of the first (generalized) mean value theorem. 3

UNIT—III

7. (a) Show that the integral

$$\int_0^{\pi/2} \sin x \log(\sin x)\,dx$$

is convergent and hence evaluate it. 5

(b) Find the total length of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$. 4

(c) Find the value of $\Gamma\left(\frac{9}{2}\right)$. 1

8. (a) Find the length of the cardioid $r = a(1-\cos\theta)$ lying outside the circle $r = a\cos\theta$. 5

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(b) Discuss the convergence of

$$\int_0^1 \frac{\sin\frac{1}{x}}{\sqrt{x}}\,dx$$

5

9. (a) Discuss the convergence of

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

4

(b) Prove that

$$B(m, n) = \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}};\quad m, n > 0$$

3

(c) Show that the volume of the catenoid formed by the revolution about the x-axis, of the area bounded by the catenary

$$y = \frac{a}{2}(e^{x/a} + e^{-x/a})$$

the y-axis, the x-axis and an ordinate is

$$\frac{1}{2}\pi a(sy + ax)$$

s being the length of the arc between $(0, a)$ and (x, y). 3

UNIT—IV

10. (a) By changing the order of integration, prove that

$$\int_0^1 dx \int_x^{1/x} \frac{y^2 dy}{(x+y)^2\sqrt{1+y^2}} = \frac{(2\sqrt{2}-1)}{2}$$ 5

(b) Obtain the Fourier series expansion of $f(x) = x\sin x$ on $[-\pi, \pi]$ and hence deduce

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots$$ 5

11. (a) Examine whether the sequence of functions $\{f_n\}$, defined by $f_n(x) = nx(1-x)^n$ is uniformly convergent in $[0, 1]$ or not. 5

(b) Let $\Sigma a_n x^n$ be a power series with sum function f and radius of convergence $R(>0)$. Then show that f is continuous, derivable and find its derivative in $(-R, R)$. 5

12. (a) Evaluate :

$$\iiint_E \frac{dxdydz}{x^2+y^2+(z-2)^2}$$

where $E: x^2+y^2+z^2=1$. 5

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(b) Give an example, with justification, of a sequence of real-valued functions which converges pointwise but is not uniformly convergent. 5

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22/03/2021

S-5/MTMH/05/20

TDP (Honours) 5th Semester Exam., 2020
(Held in 2021)

MATHEMATICS

(Honours)

FIFTH PAPER

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **two** questions from each Unit

UNIT—I

1. *(a)* State least upper bound axiom of $\mathbb{R}$. Is the set Q of all rational numbers enjoy LUB property? Justify your answer. 1½+3½=5

   (b) State and prove Heine-Borel theorem. 1+4=5

2. *(a)* Discuss the convergence of the sequence $\{nx^n\}$. 5

   (b) State Cauchy's first limit theorem for sequence. Is the converse of this theorem always true? Justify your answer. 2+3=5

(Turn Over)

13-21/**146**

3. (a) Define limit superior and limit inferior of a sequence using inequalities. Find the limit superior and limit inferior of the sequence $\{x_n\}$, if they exist, where

$$x_n = \begin{cases} 1, \text{ if } n = 1 \\ 2, \text{ if } n = 2, 4, 6, 8, \cdots \\ \text{least prime factor of } n, \text{ if } n = 3, 5, 7, 9, \cdots \end{cases}$$

1½+1½+2=5

(b) State and prove Bolzano-Weierstrass theorem for sequences. 5

UNIT—II

4. (a) Let $f : [a, b] \to \mathbb{R}$ be bounded. Show that f is Riemann integrable if and only if for each $\varepsilon > 0$, there corresponds a $\delta > 0$ such that for every partition P of $[a, b]$ with $||P|| < \delta$,

$$U(P, f) - L(P, f) < \varepsilon$$ 5

(b) Give an example of a function, with justification, which is bounded but not Riemann integrable. 2½

(c) We know that "if a function f is bounded and integrable on $[a, b]$, then $|f|$ is also bounded and integrable on $[a, b]$." Is the converse always true? Justify. 2½

5. (a) If f and g are two functions, both bounded and integrable on $[a, b]$, then prove that their product fg is also bounded and integrable on $[a, b]$. 5

(b) Define primitive of a function. Is function admitting of a primitive always continuous? Justify. 1½+3½=5

6. (a) Prove that the integral of an integrable function is continuous. 3

(b) State Weierstrass's form of second mean value theorem of integral calculus. 2

(c) If $f, g : [a, b] \to \mathbb{R}$ be bounded functions and P be any partition of $[a, b]$, then with usual notations, prove that—

(i) $L(P, f + g) \geq L(P, f) + L(P, g)$;

(ii) $U(P, f + g) \leq U(P, f) + U(P, g)$.

2½+2½=5

UNIT—III

7. (a) Show that

$$\int_0^\infty \frac{\sin x}{x} dx$$

converges but not absolutely. 5

(b) Define gamma function and discuss its convergence. 5

8. *(a)* With usual notations, show that

$$\Gamma(n)\,\Gamma\left(n+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2n-1}}\Gamma(2n),\ n>0$$ 5

(b) Show that the arc of the upper-half of the cardioid $r=a(1-\cos\theta)$ is bisected at $\theta=\frac{2\pi}{3}$. Hence find the perimeter of the curve. 5

9. *(a)* Find the volume and surface area of solid generated by revolving about y-axis that part of the astroid

$$x=a\cos^3\theta,\ y=a\sin^3\theta$$

that lies in the first quadrant. 5

(b) If $f(x)$ be continuous on $[a, b]$ and

$$\lim_{x\to a^+}(x-a)^{\mu}f(x)$$

be non-zero finite number, then prove that $\int_a^b f(x)\,dx$ converges absolutely, when $0<\mu<1$. 5

UNIT—IV

10. *(a)* If $\{f_n\}$ be a sequence of real-valued integrable functions, which converges uniformly to the function f in $[a, b]$, then prove that f is integrable in $[a, b]$ and

$$\int_a^b f(x)\,dx=\lim_{n\to\infty}\int_a^b f_n(x)\,dx$$ 5

(b) Obtain the Fourier series expansion of

$$f(x)=|\sin x| \text{ on } [-\pi,\pi]$$ 5

11. *(a)* Find the Fourier series for the function $f(x)$ defined by

$$f(x)=\begin{cases}x & \text{for } 0<x<1\\ 1-x & \text{for } 1<x<2\end{cases}$$

Hence deduce $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}$. 5

(b) Evaluate

$$\iiint_E x^{\alpha}y^{\beta}z^{\gamma}(1-x-y-z)^{\lambda}\,dxdydz$$

when $\alpha>-1$, $\beta>-1$, $\gamma>-1$, $\lambda>-1$ and E is the tetrahedron bounded by the coordinate planes and the plane $x+y+z=1$. 5

12. *(a)* By changing the order of integration, show that

$$\int_0^1 dx\int_x^{1/x}\frac{y\,dy}{(1+xy)^2(1+y^2)}=\frac{\pi-1}{4}$$ 5

(b) Find the condition so that the series

$$\sum_{n=1}^{\infty}\frac{x}{n^p+x^2n^q}$$

converges uniformly for all real x. 5

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30/12/2019

TDP (Honours) 5th Semester Exam., 2019

MATHEMATICS

(Honours)

FIFTH PAPER

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **two** questions from each Unit

UNIT—I

1. (a) Show that for any two real numbers x and y with $x < y$ $\exists$ an irrational number z s.t. $x < z < y$. 3

   (b) Let (x_n) be a real sequence s.t. $x_n \to 0$ as $n \to \infty$. Show that the sequence (s_n) defined by

$$s_n = \frac{x_1 + x_2 + \cdots + x_n}{n} \to 0 \text{ as } n \to \infty$$ 4

   (c) Prove that every compact subset of $\mathbb{R}$ (the set of real numbers) is closed. 3

2. *(a)* State Heine-Borel theorem in $\mathbb{R}$ (the set of real numbers). Let

$$F = \left\{\left(\frac{1}{n}, \frac{2}{n}\right) : n = 2, 3, 4, \ldots\right\}$$

be an open cover of (0, 1). Show that no finite sub-collection of F can cover (0, 1). Is [0, 1] compact? Justify your answer. 1½+2½+1=5

(b) State nested intervals theorem. Justify with an example that the condition of closedness of the intervals in the statement cannot be relaxed. 3

(c) Give an example of a sequence which has infinite number of sub-sequential limits. 2

3. *(a)* Prove that the sequence $\{x_n\}$, defined by

$$x_1 = \sqrt{2},\ x_{n+1} = \sqrt{2x_n},\ n \in \mathbb{N}$$

converges and find the limit. 5

(b) Show that a real sequence is convergent iff it is Cauchy. 5

UNIT—II

4. *(a)* Let f be a bounded function on $[a, b]$ and P be a partition of $[a, b]$. If P' is a refinement of P, then show that

$$L(P, f) \le L(P', f) \le U(P', f) \le U(P, f)$$ 5

(b) Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$. Show that $\frac{1}{f}$ is Riemann integrable on $[a, b]$, if $\exists\, k \in \mathbb{R}^+$ s.t. $f(x) \ge k\ \forall x \in [a, b]$. 5

5. *(a)* Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q} \cap [0, 1] \\ x^2, & \text{if } x \in (\mathbb{R} \cdot \mathbb{Q}) \cap [0, 1] \end{cases}$$

Discuss the Riemann integrability of f on [0, 1]. 5

(b) If $f : [a, b] \to \mathbb{R}$, $g : [a, b] \to \mathbb{R}$ be both integrable on $[a, b]$ and g keeps same sign over $[a, b]$, then show that there exists a number μ lies between the bounds of f in $[a, b]$ such that

$$\int_a^b f(x)\, g(x)\, dx = \mu \int_a^b g(x)\, dx$$

Further if f is continuous on $[a, b]$, then show that there exists a point ξ in $[a, b]$ such that

$$\int_a^b f(x)\, g(x)\, dx = f(\xi) \int_a^b g(x)\, dx$$ 3½+1½=5

6. *(a)* Let $f: [a, b] \to \mathbb{R}$ be continuous except only at finite number of points on $[a, b]$. Prove that f is Riemann integrable on $[a, b]$. 5

   (b) If f is a bounded function on $[a, b]$, then prove that corresponding to every $\varepsilon > 0$, there exists a $\delta > 0$, such that $U(P, f) < \overline{\int_a^b} f(x)\, dx + \varepsilon$, for every partition P of $[a, b]$ with $\|P\| < \delta$. 5

UNIT—III

7. *(a)* Discuss the convergence of

$$\int_0^{\frac{\pi}{2}} x^m \operatorname{cosec}^n x\, dx$$ 5

   (b) Find the total length of the astroid

$$x^{2/3} + y^{2/3} = a^{2/3}$$ 5

8. *(a)* Find the whole length of the loop of the curve

$$3ay^2 = x(x-a)^2,\ a > 0$$ 5

   (b) Discuss the convergence of

$$\int_0^1 \frac{\sin \frac{1}{x}}{\sqrt{x}}\, dx$$ 3

   (c) By using definition of β-function, show that the β-function is symmetric in two variables, i.e., $\beta(m, n) = \beta(n, m)$. 2

9. *(a)* Find out the volume of the solid bounded by infinite spindle-shape surface generated by revolving the curve $y = \frac{1}{1+x^2}$ about its asymptote. 5

   (b) Evaluate

$$\int_0^t x^{\alpha+k-1} (t-k)^{\beta+k-1}\, dx$$

and find its value, when $\alpha = \beta = \frac{1}{2}$. 5

UNIT—IV

10. *(a)* If a sequence $\{f_n\}$ of real-valued functions converges uniformly in $X \subseteq \mathbb{R}$ to a function f and x_0 is a limit point of X such that

$$\lim_{x \to x_0} f_n(x) = a_n, \; n = 1, 2, 3, \cdots$$

then prove that—

(i) $\{a_n\}$ converges;

(ii) $\lim_{x \to x_0} f(x)$ exists and equal to $\lim_{n \to \infty} a_n$. 2+3=5

(b) Obtain the Fourier series expansion of $f(X) = X \sin X$ on $[-\pi, \pi]$ and hence deduce

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} - \cdots$$ 5

11. *(a)* Show that the sequence of functions (f_n), defined by

$$f_n(X) = \frac{nx}{1 + n^2 x^2}, \; 0 \le x \le 1$$

is not uniformly convergent on [0, 1]. 5

(b) Assuming the validity of differentiation under the integral sign, show that

$$\int_0^{\pi} \frac{\log(1 + \sin\alpha \cos x)}{\cos x} dx = \pi\alpha$$ 5

12. *(a)* Show that a power series $\sum a_n x^n$ can be integrated term-by-term within the interval of convergence and also show that the original series and the series obtained after term-by-term integration have same radius of convergence. 5

(b) By changing the order of integration, prove that

$$\int_0^1 dx \int_x^{\frac{1}{x}} \frac{y^2 \, dy}{(x+y)^2 \sqrt{1+y^2}} = \frac{2\sqrt{2} - 1}{2}$$

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01/12/18

TDP (Honours) 5th Semester Exam., 2018

MATHEMATICS

(Honours)

FIFTH PAPER

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **two** questions from each Unit

UNIT—I

1. *(a)* Show that $\mathbb{Z}$ is neither bounded above nor below. 4

(b) Discuss the convergence of the sequence (x_n) given by

$$x_n = \frac{1}{\lfloor 1} - \frac{1}{\lfloor 2} + \cdots + \frac{(-1)^{n+1}}{\lfloor n}$$ 3

(c) If $-1 < r \le 1$, then prove that the sequence $\{r^n\}$ is convergent. 3

(Turn Over)

2. (a) Test the behaviour of the sequence

$$\left\{\left(1+\frac{1}{n}\right)^n\right\}$$ 5

(b) Prove that every bounded sequence of real numbers has a convergent subsequence. 5

3. (a) Prove that arbitrary intersection of closed sets is closed. Is the result true for arbitrary union? Justify your answer. 3+2=5

(b) State and prove Cauchy criterion for convergence of real sequences. 1+4=5

UNIT—II

4. (a) Let $f:[a, b]\to\mathbb{R}$ be bounded. Show that SKM-396 f is Riemann integrable if and only if for each $\varepsilon>0$, $\exists$ a partition P of $[a, b]$ such that $U(P, f)-L(P, f)<\varepsilon$. 5

(b) Let $f:[a, b]\to\mathbb{R}$ be integrable on $[a, b]$. Then show that $|f|$ is also integrable on $[a, b]$. Is the converse true? Justify your answer. 4+1=5

(Continued)

SM- 298

5. (a) Discuss the Riemann integrability of the function $f:[0, 1]\to\mathbb{R}$ given by

$$f(x)=\begin{cases}\frac{1}{q}, & \text{when } x=\frac{p}{q},\ p, q\in\mathbb{N},\ (p, q)=1\\ 0, & \text{when } x \text{ is irrational or zero}\end{cases}$$ 5

(b) State and prove first mean value theorem of integral calculus. 5

6. (a) State Bonnet's form of second mean value theorem of integral calculus. Applying this theorem, show that

$$\left|\int_a^b \frac{\sin x}{x}\,dx\right|\le\frac{2}{a}$$

where $0<a<b<\infty$. 5

(b) If a function $f:[a, b]\to\mathbb{R}$ be integrable on $[a, b]$, then prove that the function F defined by $F(x)=\int_a^x f(t)\,dt,\ x\in[a, b]$ is differentiable at any point $c\in[a, b]$ at which f is continuous and $F'(c)=f(c)$. 5

UNIT—III

7. (a) Discuss the convergence of

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$ 5

(b) Show that $\int_0^1 \log\sqrt{x}\,dx$ is convergent and hence evaluate it. 5

MA-323

(Turn Over)

8. *(a)* When n is a positive integer, show that

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{(2n)!}{2^{2n}\,n!}\sqrt{\pi}$$ 5

(b) Find the volume and surface area of the solid generated by revolving one arch of the cycloid $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$ about its base. 5

9. *(a)* Find the perimeter of the loop of the curve

$$9ay^2=(x-2a)(x-5a)^2,\ a>0$$ 5

(b) Discuss the convergence of the integral

$$\int_0^\infty x^{-\frac{1}{2}}e^{-x}\,dx$$ 5

UNIT—IV

10. *(a)* Examine whether the sequence $\{f_n\}$, defined by $f_n(x)=nx(1-x)^n$ is uniformly convergent in $[0,\ 1]$ or not. 5

(b) Obtain the Fourier series expansion of the function $f(X)=X\sin X$ on $[-\pi,\ \pi]$ and hence deduce

$$\frac{\pi}{4}=\frac{1}{2}+\frac{1}{1\cdot3}-\frac{1}{3\cdot5}+\frac{1}{5\cdot7}-\cdots$$ 5

SM - 2018

11. *(a)* Apply Parseval's identity to the function $f(x)=x,\ -\pi\le x\le\pi$ and deduce that

$$\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}+\cdots$$ 5

(b) By changing the order of integration, prove that

$$\int_0^1 dx\int_0^{\sqrt{1-x^2}}\frac{dy}{(1+e^y)\sqrt{1-x^2-y^2}}=\frac{\pi}{2}\log\left(\frac{2e}{1+e}\right)$$ 5

12. *(a)* Show that

$$\int_0^\pi\frac{\log(1+a\cos x)}{\cos x}dx=\pi\sin^{-1}a,\ |a|<1$$ 5

(b) Let $\sum a_n x^n$ be a power series with sum function f and radius of convergence $R(>0)$. Then show that f is continuous, derivable and $f'(x)=\Sigma n a_n x^{n-1}$, $x\in(-R,\ R)$. 5

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01/12/2077

S–5/MTMH/05/17

TDP (Honours) 5th Semester Exam., 2017

MATHEMATICS

(Honours)

FIFTH PAPER

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **two** questions from each Unit

UNIT—I

1. *(a)* State and prove Bolzano-Weierstrass theorem for infinite set in $\mathbb{R}$ (the set of real numbers). 4

   (b) Show that the set $\mathbb{R}\setminus\mathbb{Q}$ is dense in $\mathbb{R}$. 3

   (c) Show that a monotonically increasing sequence is convergent if and only if it is bounded above. 3

2. *(a)* State and prove the nested interval theorem on $\mathbb{R}$. 5

   (b) State and prove Cauchy's first limit theorem. 5

SM- 2017

3. *(a)* Prove that $\phi \neq S \leq \mathbb{R}$ is compact if and only if it is closed and bounded. 5

(b) Prove that the intersection of a finite number of open sets is an open set. Is the intersection of arbitrary family of open sets always open? Justify your answer. 3+2=5

UNIT—II

4. *(a)* If f is a bounded function on $[a, b]$, then prove that corresponding to every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$U(P, f) < \int_a^b f(x)\,dx + \varepsilon$$

for every partition P of $[a, b]$ with $||P|| < \delta$. 5

(b) Discuss the Riemann integrability of the function $f : [0, \frac{\pi}{4}] \to \mathbb{R}$, given by

$$f(x) = \begin{cases} \cos x, & x \in \mathbb{Q} \\ \sin x, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$ 5

5. *(a)* Let $f : [a, b] \to \mathbb{R}$ be continuous at $c \in [a, b]$ and $F(x) = \int_a^x f(t)\,dt, \ \forall x \in [a, b]$.

Then prove that F is derivable at c and $F'(c) = f(c)$. 4

(b) Let $f : [a, b] \to \mathbb{R}$ be continuous except for finitely many points in $[a, b]$. Show that f is Riemann integrable on $[a, b]$. 4

 SM- 2017

(c) Let $f, g : [-1, 1] \to \mathbb{R}$ be given by $f(x) = \text{sgn}(x)$ and $g(x) = [x]$. Is f Riemann integrable? Is g Riemann integrable? What can be said about the Riemann integrability of $f + g$? 2

6. *(a)* State and prove Weierstrass form of second mean value theorem of integral calculus. 5

(b) A function $f : [0, 1] \to \mathbb{R}$ is defined by

$$f(x) = \frac{1}{a^{r-2}}$$

where $\frac{1}{a^r} < x \leq \frac{1}{a^{r-1}}$, $r = 1, 2, 3, \ldots$ and $a > 2$. Is f Riemann integrable on $[0, 1]$? If so, find the value of $\int_0^1 f(x)\,dx$. 5

UNIT—III

7. *(a)* Show that the integral

SNTMDR p. 16.3, (2)

$$\int_0^{\pi/2} \sin x \log(\sin x)\,dx$$

is convergent and hence evaluate it. 5

(b) Discuss the convergence of the gamma function. 5

8. *(a)* Find the length of the cardioid $r = a(1 - \cos\theta)$ lying outside the circle $r = a\cos\theta$. 5

(b) Find the volume and surface area of the solid generated by revolving the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ about its base. 5

9. *(a)* Find the volume and surface area of the solid generated by revolving the cardioid $r = a(1 - \cos\theta)$ about the initial line. 5

(b) If $f(x)$ be continuous on $(a, b]$ and

$$\lim_{x \to a^+} (x-a)^{\mu} f(x)$$

be non-zero finite number, then prove that $\int_a^b f(x)\,dx$ converges absolutely, when $0 < \mu < 1$. 5

UNIT—IV

10. *(a)* Obtain the Fourier series, of the function $f(x)$ in the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$, defined by

$$f(x) = \begin{cases} x - [x] - \frac{1}{2}, & \text{when } x \text{ is not an integer} \\ 0 \quad , & \text{otherwise} \end{cases}$$ 5

(b) Using appropriate substitution, evaluate

$$\int_{x=0}^{1} \int_{y=0}^{1-x} e^{\frac{y}{x+y}} \, dy\, dx$$ 5

11. *(a)* X – 27 By changing the order of integration, show that

$$\int_0^1 \int_x^{1/x} \frac{y\,dy}{(1+xy)^2(1+y^2)} = \frac{\pi - 1}{4}$$ 5

(b) Let $f_n : (a, b) \to \mathbb{R}$ be differentiable. Assume $f,\ g : (a, b) \to \mathbb{R}$ s.t. $f_n \to f$ and $f_n' \to g$ uniformly. Then S.T. f is differentiable and $f' = g$ on (a, b). 5

12. *(a)* Evaluate

$$\iiint_E \frac{dx\,dy\,dz}{x^2 + y^2 + (z-2)^2}$$

where $E : x^2 + y^2 + z^2 = 1.$ 5

(b) Give an example, with justification, of a sequence of real-valued functions which converges pointwise but is not uniformly convergent. 5

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AP 08/12/2016

S–5/MTMH/05/16

TDP (Honours) 5th Semester Exam., 2016

MATHEMATICS

(Honours)

FIFTH PAPER

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **two** questions from each Unit

UNIT—I

1. *(a)* Show that an upper bound u of $\phi \neq S \subseteq \mathbb{R}$ is the supremum of S if and only if $\forall \varepsilon > 0 \, \exists S_\varepsilon \in S$ such that $u - \varepsilon < S_\varepsilon$. 3

K-8

(b) Show that every convergent sequence is bounded. Is the converse true? Justify your answer. 3

K-62

(c) Prove that an open interval (a, b) in $\mathbb{R}$ (the set of real numbers) is not compact. 4

K-47

2. *(a)* State and prove the Archimedean property of $\mathbb{R}$ (the set of real numbers). 4

K-16

(b) Discuss the convergence of the sequence $\{x_n\}$ defined by $x_n = \left(1+\frac{1}{n}\right)^n$, for $n \in N$. 3

K-72

(c) Examine whether the set $[0, \infty)$ is compact in $\mathbb{R}$ or not by only using the definition of compact sets. 3

K-46

3. (a) Show that the set of rationals forms a dense subset of $\mathbb{R}$. 5

K-44

(b) Let A and B be two non-empty subsets of $\mathbb{R}$ (the set of real numbers) and $C = \{x+y : x \in A \text{ and } y \in B\}$. If each of A and B has supremum, then show that C has supremum and $\sup C = \sup A + \sup B$. 5

K-22

UNIT—II

4. (a) Let $f:[0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0, & \text{when } x \text{ is irrational or zero} \\ \frac{1}{q}, & \text{when } x = \frac{p}{q}, \text{ where } p, q \in \mathbb{N} : (p, q) = 1 \end{cases}$$

Verify whether f is Riemann integrable or not in $[0, 1]$. 5

(b) Let $f:[a, b] \to \mathbb{R}$ be bounded. Show that f is Riemann integrable if and only if for each $\varepsilon > 0$ $\exists$ a partition P of $[a, b]$ such that

$$U(p, f) - L(p, f) < \varepsilon.$$ 5

SM- 2016

5. (a) A function f is bounded and integrable in $[a, b]$, then $|f|$ is also bounded and integrable on $[a, b]$. Is the converse true in general? Justify your answer. 4

(b) Let $f:[-2, 2] \to \mathbb{R}$ be defined by $f(X) = [X]$. Discuss the Riemann integrability of f and if f is Riemann integrable, find the value of the integral. 3

(c) If $f:[a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$, then prove that the function F defined by

$$F(x) = \int_a^x f(t)\, dt$$

is continuous. on $[a, b]$. 3

6. (a) Suppose $f:[a, b] \to \mathbb{R}$ and $g:[a, b] \to \mathbb{R}$ be such that

(i) f is continuous on $[a, b]$ and g is Riemann integrable on $[a, b]$

(ii) g has no root in $[a, b]$.

Then show that $\exists\ \xi \varepsilon [a, b]$ such that

$$\int_a^b f(x)\, g(x)\, dx = f(\xi)\int_a^b g(x)\, dx$$ 5

(b) State Weierstrass form of second mean value theorem of integral calculus. Applying this theorem, show that

$$\left|\int_a^b \frac{\sin x}{x}\, dx\right| \le \frac{4}{a}, \text{ for } 0 < a < b < \infty$$ 5

 SM - 2016

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UNIT—III

7. (a) Show that $\int_0^\infty \frac{\sin x}{x} dx$ is convergent but not absolutely. 5

(b) Discuss the convergence of the integral

$$\int_0^1 x^{m-1}(1-x)^{n-1} dx$$ 3

(c) Find the length of the parabola $y^2 = 16x$ measured from the vertex to one extremity of the latus rectum. 2

8. (a) Discuss the convergence of the beta function. 5

(b) Find the surface area of the solid generated by revolving the cycloid $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$ about the x-axis. 5

9. (a) Show that the arc of the upper half of the cardioid $r = a(1-\cos\theta)$ is bisected at $\theta = \frac{2\pi}{3}$. Hence find the perimeter of the cardioid. 5

SM+MDR P-20.21 (b) Show that $\sqrt{\pi}\,\Gamma(2n) = 2^{2n-1}\Gamma(n)\Gamma\left(n+\frac{1}{2}\right)$ for $n > 0$. 5

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UNIT—IV

10. (a) K-31 Let $\{f_n\}$ be a sequence of real valued continuous functions defined on $[a, b]$. If $\{f_n\}$ converges uniformly to the limit function f on $[a, b]$, then prove that f is continuous on $[a, b]$. 5

(b) Show that

$$\iint_R \frac{\sqrt{a^2b^2 - b^2x^2 - a^2y^2}}{\sqrt{a^2b^2 + b^2x^2 + a^2y^2}} dx\, dy = \pi ab\left(\frac{\pi}{2} - 1\right)$$

where the field of integration R varies over the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ 5

11. (a) Let $\sum_n a_n X^n$ be a power series with radius of convergence R. Prove that the series is uniformly convergent on $[-s, s]$ where $0 < s < R$. 4 SKM-531

(b) By changing the order of integration, prove that

K-28 $$\int_0^1 dx \int_x^{1/x} \frac{y^2 dy}{(x+y)^2\sqrt{1+y^2}} = \frac{(2\sqrt{2}-1)}{2}$$ 5

(c) State Parseval's identity. 1

5M 2016

U.C → 5/10

12. *(a)* The function f is defined by $f(x) = 1 + 2 \cdot 4x + 3 \cdot 4^2 x^2 + 4 \cdot 4^3 x^3$

$$+ \cdots + n \cdot 4^{n-1} x^{n-1} + \cdots$$

Show that f is continuous on $(-\frac{1}{4}, \frac{1}{4})$.

Evaluate $\int_0^{1/8} f(x)\, dx$. 4

(b) Let $D \subset \mathbb{R}$ and for each $n \in \mathbb{N}$ $f_n : D \to \mathbb{R}$ be bounded on D. If the sequence $\{f_n\}$ is uniformly convergent on D, then show that the limit function f is bounded on D. 3

(c) Evaluate :

$$\int_0^{\infty} e^{-t^2} \cos xt\, dt$$ 3

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